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ON THE MAXIMAL ERGODIC THEOREM FOR CERTAIN SUBSETS OF THE INTEGERS

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ABSTRACT

It is shown that the set of squares $\{n^2 \mid n = 1, 2, \dots\}$ or, more generally, sets $\{n^t \mid n = 1, 2, \dots\}$, t a positive integer, satisfies the pointwise ergodic theorem for L^2-functions. This gives an affirmative answer to a problem considered by A. Bellow [Be] and H. Furstenberg [Fu]. The previous result extends to polynomial sets $\{p(n) \mid n = 1, 2, \dots\}$ and systems of commuting transformations. We also state density conditions for random sets of integers in order to be "good sequences" for L^p-functions, $p > 1$.

1. Introduction

Let (Ω, μ, T) be a dynamical system and N a subset of the positive integers. For bounded measurable functions f on Ω, consider the "maximal function" of the ergodic averages with respect to N

$$\mathcal{M}_N f = \sup_{j \geq 1} |\mathcal{A}_{\Lambda_j} f| \tag{1.1}$$

where

$$\mathcal{A}_\Lambda f = \frac{1}{|\Lambda|} \sum_{n \in \Lambda} T^n f \quad \text{and} \quad \Lambda_j = N \cap [0, j].$$

The purpose of this paper is to prove for certain arithmetic sets N the *a priori* L^2-bound on $\mathcal{M}_N$. Considering for (Ω, μ, T) the shift on $\mathbf{Z}$ (or on a cyclic group $\mathbf{Z}_K$), $\mathcal{M}_N$ becomes the maximal function corresponding to a sequence of convolution operators. As such, we will prove boundedness properties using Fourier transform methods. Of course, this is essentially an L^2-theory and

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does not enable to obtain weak-type bounds on L^1, for instance (thus the full analogue of Birkhoff's theorem). There is a variety of results that one can obtain by this method, the main request being to have enough information about the Fourier transform of the underlying kernels. In the applications listed below, these Fourier transforms appear as Weyl exponential sums and have been well studied in Number Theory. We will use results from [S], [Vaug], and [Vin].

THEOREM 1. *If $N = \{n^2 \mid n \in \mathbf{N}\}$, then $\mathcal{M}_N$ is L^2-bounded, i.e., there is the following inequality (a priori for bounded measurable functions):*

$$\| \mathcal{M}_n f \|_{L^2(\mu)} \leqq C \| f \|_{L^2(\mu)}. \tag{1.2}$$

THEOREM 2. *The analogue of Theorem 1 for $N = \{n^t \mid n \in \mathbf{N}\}$, t a positive integer.*

The proof of Theorem 1 is more elementary and we tried to make the argument more self-contained. The relevant exponential sums for $t = 2$ are Gauss sums which are easier to handle. We will make use of an estimate from Sarközy's paper [S] (see Lemma 4.1).

In proving Theorem 2, we will use some facts related to exponential sums which appear in solving the Waring problem. Our discussion is based on the references [Vaug] and [Vin].

Theorem 2 can be extended to polynomial functions $p(n)$, p with integer coefficients, using the same method as for powers $p(n) = n^t$. Possibly the L^2-estimates appearing here can be combined with certain interpolation methods to get the maximal inequality on L^p for $p > 1$. This will be investigated elsewhere.†

There is the following corollary of the equidistribution (mod 1) of $\{n^t \alpha\}$ for α an irrational number.

COROLLARY 3. *$(1/n) \sum_{m=0}^{n-1} f(x + m^t \alpha) \to \int_\pi f(x)dx$ a.s. for $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ and f a bounded measurable function on $\mathbf{T} = \mathbf{R}/\mathbf{Z}$.*

Observe that the equidistribution results only imply convergence in the mean. There is also the following consequence related to results in Marstrand's paper [M].††

† See the forthcoming paper [B2] in this respect.

†† The results presented in this paper remain valid for positive isometries.

COROLLARY 4. *$(1/n)\sum_{m=0}^{n-1} f(2^{(m^t)}x) \to \int f(x)dx$ a.s., for f bounded measurable on* $\mathbf{T}$, *generalizing the Riesz–Raikov result for $t = 1$.*

Corollary 3 has the following generalization for arbitrary measure preserving transformations.

THEOREM 5. *Let (Ω, μ, T) be a dynamical system and $t \geqq 1$ an integer. Then, for $f \in L^2(\Omega, \mu)$,*

$$\mathscr{A}_n f = \frac{1}{n} \sum_{m=1}^{n} T^{(m^t)} f$$

converges a.e. for $n \to \infty$.

The same statement holds replacing n^t by $p(n)$ an arbitrary polynomial with integer coefficients.

As for Theorem 2, the proof of Theorem 5 is only worked out for powers $p(n) = n^t$. The argument for polynomials is almost identical and left to the reader. (The relevant exponential sums for a polynomial $p(n)$ of degree t are at least as good as in the case $p(n) = n^t$, discussed in Section 6 of this paper.) An alternative approach is presented in Appendix 2.

The main problem here is due to the fact that, in general, there is no natural dense set of functions for which the a.s. convergence holds. Theorem 5 is thus not an immediate consequence of Theorem 2. However, we will use the maximal function inequality given by Theorem 2 and the elements of the proof. Observe indeed that the L^2-boundedness of $\mathscr{M} f = \sup_{n \geqq 1} |\mathscr{A}_n f|$ reduces the problem to L^∞-bounded functions.

The main results of this paper were announced in [B1].

2. Reduction to the shift

In the context of the maximal function problem, the case of a general dynamical system (Ω, μ, T) is equivalent to the shift S acting on a cyclic group $\mathbf{Z}_K = \mathbf{Z}/K\mathbf{Z}$ or on the integers $\mathbf{Z}$. (In the $\mathbf{Z}$-case, we may take functions with a finite support in proving *a priori* inequalities.) Let $x \in \Omega$ be a fixed point and define $\tilde{f}$ on $\mathbf{Z}$ by

$$\tilde{f}(j) = f(T^j x) = T^j f(x) \tag{2.1}$$

where f is a bounded measurable function on Ω.

Then for $\Lambda \subset \mathbf{Z}$

$$\mathscr{A}_\Lambda \tilde{f}(j) = \frac{1}{|\Lambda|} \sum_{k \in \Lambda} \tilde{f}(j+k) = \frac{1}{|\Lambda|} \sum_{k \in \Lambda} T^k f(T^j x) = \mathscr{A}_\Lambda f(T^j x)$$

and thus

$$\mathscr{M}_{N_0} \tilde{f}(j) = \mathscr{M}_{N_0} f(T^j x) \tag{2.2}$$

for any initial segment $N_0 = N \cap [0, R]$ of the set N.

Assume now $1 < p < \infty$ and that we have an inequality relative to $(\mathbf{Z}, S)$

$$\| \mathscr{M}_N \varphi \|_p \leqq B \| \varphi \|_p. \tag{2.3}$$

Put $\varphi = \tilde{f}\,|_{[|j| \leqq J]}$. Since clearly $\mathscr{M}_{N_0}\tilde{f}(j) = \mathscr{M}_{N_0}\varphi(j)$ if $|j| < J - R$, it follows from (2.3) that

$$\sum_{|j| < J-R} |\mathscr{M}_{N_0} \tilde{f}(j)|^p \leqq B^p \sum |\varphi(j)|^p,$$

hence from (2.1), (2.2)

$$\frac{1}{J} \sum_{|j| < J-R} |\mathscr{M}_{N_0} f(T^j x)|^p \leqq B^p \frac{1}{J} \sum_{|j| \leqq J} |f(T^j x)|^p.$$

Since this inequality is valid for all $x \in \Omega$, integration and using the fact that T is measure preserving yields

$$\frac{1}{J}(J - R) \| \mathscr{M}_{N_0} f \|_p^p \leqq B^p \| f \|_p^p,$$

and letting $J \to \infty$

$$\| \mathscr{M}_{N_0} f \|_p \leqq B \| f \|_p.$$

Since N_0 is an arbitrary finite segment of N, this completes the proof.

REMARKS. (1) Previous argument yields also a proof of the weak-type inequality for the maximal function in the context of Birkhoff's theorem.

(2) The generalization to positive isometries is straightforward.

3. Fourier transform estimate of certain maximal functions

For $\alpha \in \mathbf{T}$, the Fourier transform $\hat{f}(\alpha)$ of a (finitely supported) function f on $\mathbf{Z}$ is given by

$$\hat{f}(\alpha) = \sum_{j \in \mathbf{Z}} f(j) e^{-2\pi i j \alpha}$$

and by Parsaval's identity

$$\sum_{j\in\mathbf{Z}} |f(j)|^2 = \int_0^1 |\hat{f}(\alpha)|^2 d\alpha.$$

If T is the shift on $\mathbf{Z}$, then

$$\frac{1}{n+1}\sum_{m=0}^{n} T^{m^t} f = f * K_n \quad \text{where } K_n = \frac{1}{n+1}\sum_{m=0}^{n} \delta_{\{m^t\}}.$$

Thus

$$(f * K_n)(x) = \int_0^1 \hat{f}(\alpha)\hat{K}_n(\alpha) e^{2\pi i\alpha x} d\alpha$$

where $\hat{K}_n(\alpha)$ is given by the exponential sum

$$\hat{K}_n(\alpha) = \frac{1}{n+1}\sum_{m=0}^{n} e^{2\pi i m^t \alpha}.$$

Clearly, in evaluating the maximal operator, it suffices to consider diadic values of n, i.e., $n \in Z \equiv \{2^k \mid k = 1, 2, \ldots\}$. The advantage of considering Fourier transforms lies in the following consideration. Assume we obtained for each n an approximation $\hat{L}_n$ of $\hat{K}_n$ in the sense that

$$\| \hat{K}_n - \hat{L}_n \|_\infty < (\log n)^{-\sigma}, \qquad \sigma > \tfrac{1}{2}. \tag{3.1}$$

We then have

$$\sup_{n\in Z} |f * K_n| \leqq \sup_Z |f * L_n| + \Big(\sum_{n\in Z} |f * (K_n - L_n)|^2\Big)^{1/2},$$

$$\Big\| \sup_Z |f * K_n| \Big\|_2 \leqq \Big\| \sup_Z |f * L_n| \Big\|_2 + \Big(\sum_{n\in Z} \| \hat{K}_n - \hat{L}_n \|_\infty^2\Big)^{1/2} \| f \|_2,$$

where, by (3.1),

$$\Big(\sum_{n\in Z} \| \hat{K}_n - \hat{L}_n \|_\infty^2\Big)^{1/2} \leqq \Big(\sum_{j=1}^{\infty} j^{-2\sigma}\Big)^{1/2} < \infty.$$

Essentially speaking, the $\hat{L}_n$ are obtained by restricting $\hat{K}_n$ to a system of major arcs, where it has a "nice" behaviour. The following lemma will be useful in evaluating the major arc contribution.

Let $0 \leqq \varphi \leqq 1$ be a smooth bumpfunction vanishing outside a τ-neighborhood of 0. Assume $k \in L^1(\mathbf{R})$ satisfies a maximal inequality

$$\left\| \sup_{t>0} | f * k_t | \right\|_{L^2(\mathbf{R})} \leqq C(k) \| f \|_{L^2(\mathbf{R})} \quad \text{where } k_t(x) = \frac{1}{t} k\left(\frac{x}{t}\right). \tag{3.2}$$

Let $\mathscr{R}$ be a set of rational points $\theta \in [0, 1]$ which can be given a common denominator Q satisfying

$$Q\tau \ll 1. \tag{3.3}$$

For f a (finitely supported) function f on $\mathbf{Z}$, define

$$A_t f(x) = \sum_{\theta \in \mathscr{R}} \int \hat{k}(t(\alpha - \theta)) \hat{f}(\alpha) c^{2\pi i \alpha x} \varphi(\alpha - \theta) d\alpha \tag{3.4}$$

and the maximal operator

$$Mf = \sup_{t>0} | A_t f |.$$

Then

$$\| Mf \|_{l^2(\mathbf{Z})} \leqq 4c(k) \| f \|_{l^2(\mathbf{Z})}. \tag{3.5}$$

PROOF. Rewrite $A_t f$ as

$$A_t f(x) = \int_{-\infty}^{\infty} \hat{k}(t\beta) \varphi(\beta) \left[\sum_{\theta \in \mathscr{R}} \hat{f}(\theta + \beta) e^{2\pi i (\theta + \beta) x} \right] d\beta$$

and put $x = Qy + z$ where $y \in \mathbf{Z}$, $z \in \{0, 1, \ldots, Q-1\}$. Thus by hypothesis on $\mathscr{R}$ and Q

$$A_t f(x) = \int \hat{k}(t\beta) \varphi(\beta) F(z, \beta) e^{2\pi i \beta Q y} d\beta \tag{3.6}$$

where

$$F(z, \beta) = \sum_{\theta \in \mathscr{R}} \hat{f}(\theta + \beta) e^{2\pi i (\theta + \beta) z}.$$

We have

$$\| Mf \|^2_{l^2(\mathbf{Z})} = \sum_{z=0}^{Q-1} \left\| \sup_{t>0} \left| \int \hat{k}(t\beta) \varphi(\beta) F(z, \beta) e^{2\pi i \beta Q y} d\beta \right| \right\|^2_{l^2(\mathbf{Z}, dy)} \tag{3.7}$$

and evaluate for fixed z the corresponding term in (3.7). The purpose of what follows is to replace the $l^2(\mathbf{Z}, dy)$ and $L^2(\mathbf{R})$ norms.

Denote ζ the best constant in the *a priori* inequality

$$\left\| \sup_{t>0} \left| \int \hat{k}(t\beta)\varphi(\beta)F(\beta)e^{2\pi iQ\beta y}d\beta \right| \right\|_{l^2(\mathbf{Z})} \leqq \zeta \left(\int |F(\beta)|^2 \varphi(\beta)^2 d\beta \right)^{1/2}. \tag{3.8}$$

For $0 \leqq u \leqq 1$, estimate the left member of (3.8) as

$$\begin{aligned} &\left\| \sup_{t>0} \left| \int \hat{k}(t\beta)\varphi(\beta)F(\beta)e^{2\pi iQ\beta(y+u)}d\beta \right| \right\|_{l^2(\mathbf{Z})} \\ &\quad + \left\| \sup_{t>0} \left| \int \hat{k}(t\beta)\varphi(\beta)F(\beta)(e^{2\pi iQ\beta u} - 1)e^{2\pi iQ\beta y}d\beta \right| \right\|_{l^2(\mathbf{Z})} \end{aligned} \tag{3.9}$$

and use (3.8) to evaluate the second term in (3.9) as

$$\zeta \left(\int |F(\beta)|^2 |e^{2\pi iQ\beta u} - 1|^2 \varphi(\beta)^2 d\beta \right)^{1/2} \leqq C\zeta(Q\tau) \left(\int |F(\beta)|^2 \varphi(\beta)^2 d\beta \right)^{1/2}. \tag{3.10}$$

Integrating the first term in (3.9) with respect to $u \in [0, 1]$, there follows from (3.2) and a change of variable $y' = Qy$ the estimate

$$\left\| \sup_{t>0} \left| \int \hat{k}(t\beta)\varphi(\beta)F(\beta)e^{2\pi i\beta Qy}d\beta \right| \right\|_{L^2(\mathbf{R})} \leqq \frac{C(k)}{Q^{1/2}} \left(\int |F(\beta)|^2 \varphi(\beta)^2 d\beta \right)^{1/2}. \tag{3.11}$$

Consequently, by (3.10), (3.11) and (3.3)

$$\zeta \leqq C(Q\tau)\zeta + \frac{C(k)}{Q^{1/2}} \Rightarrow \zeta \leqq 2\frac{C(k)}{Q^{1/2}}. \tag{3.12}$$

Applying (3.8), (3.12) to (3.7) gives

$$\begin{aligned} \| Mf \|_2^2 &\leqq 4\frac{C(k)^2}{Q} \int \varphi(\beta)^2 \sum_{z=0}^{Q-1} \left| \sum_{\theta \in \mathcal{R}} \hat{f}(\theta + \beta)e^{2\pi i(\theta+\beta)z} \right|^2 d\beta \\ &= 4C(k)^2 \int \varphi(\beta)^2 \sum_{\theta \in \mathcal{R}} |\hat{f}(\theta + \beta)|^2 d\beta \\ &\leqq 4C(k)^2 \left(\int_{-1}^{2} |\hat{f}(\alpha)|^2 d\alpha \right) \end{aligned}$$

using the fact that, by (3.3), the functions φ_θ ($\theta \in \mathcal{R}$) are disjointly supported.

Thus $\| Mf \|_2 \leqq 4C(k) \| f \|_2$, proving (3.4).

4. The behaviour of Gauss sums

In this section, consider the case $t = 2$ where $\hat{K}_n(\alpha) = (1/n)\sum_{m=0}^{n-1} e^{2\pi i m^2\alpha}$. The purpose of what follows is to provide the substitute $\hat{L}_n(\alpha)$ of $\hat{K}_n(\alpha)$ by studying the behavior of $\hat{K}_n(\alpha)$ on a suitable system of major arcs. We are giving some details, rather than invoking directly [Vaug] (Lemma 2.7). The following lemma appears in [S].

LEMMA 4.1. *Let α be a real number and $a, q \geqq 1$ integers satisfying $(a, q) = 1$ and $|\alpha - a/q| < 1/q^2$. Then*

$$\left|\sum_{m=0}^{n} e^{2\pi i m^2\alpha}\right| \leqq C\left\{\frac{n}{\sqrt{q}} + (n\log q)^{1/2} + (q\log q)^{1/2}\right\}.$$

Letting for $(a, q) = 1$

$$S(q, a) = \frac{1}{q}\sum_{r=0}^{q-1} e^{2\pi i r^2 a/q} \qquad \text{(Gauss sum)}$$

we also have that

LEMMA 4.2. $|S(q, a)| < Cq^{-1/2}$.

LEMMA 4.3. *Assume $1 \leqq a \leqq q < n^{\upsilon}$ ($\upsilon = \frac{1}{100}$ say), $(a, q) = 1$ and*

$$\alpha \in \mathcal{M}(q, a) \equiv \{\alpha \in \Pi; |\alpha - a/q| < 1/n^{2-\upsilon}\}.$$

Then

$$\hat{K}_n(\alpha) = S(q, a)v_n(\alpha - a/q) + O(n^{-1/2}) \tag{4.4}$$

where

$$v_n(\beta) = \int_0^1 e^{2\pi i n^2 x^2 \beta} dx. \tag{4.5}$$

If $|\alpha| < 1/n^{2-\upsilon}$, then $\hat{K}_n(\alpha) = v_n(\alpha) + O(n^{-1/2})$.

PROOF. For $1 \leqq m \leqq n$, $m = dq + r$ ($0 \leqq r < q$) and $\beta = \alpha - a/q$,

$$m^2\alpha = (d^2q^2 + 2rdq + r^2)\left(\frac{a}{q} + \beta\right) \in \mathbf{Z} + r^2 a/q + d^2q^2\beta + O(n^{2\upsilon - 1}).$$

Hence

$$\hat{K}_n(\alpha) = \left(\frac{1}{q}\sum_{r=0}^{q-1} e^{2\pi i(a/q)r^2}\right)\left(\frac{q}{n}\sum_{d=0}^{n/q} e^{2\pi i d^2 q^2 \beta}\right) + O(q/n) + O(n^{2\upsilon-1}).$$

Since for $d \leqq \lambda < d+1$, $|e^{2\pi i\lambda^2 q^2\beta} - e^{2\pi i d^2 q^2 \beta}| < Cn^{-1+\upsilon}$,

$$\hat{K}_n(\alpha) = S(q,a)\left(\frac{q}{n}\int_0^{n/q} e^{2\pi i q^2\beta x^2}dx\right) + O(n^{-1/2})$$

implying (4.4).

Fix $0<\rho<1$ and let $0 \leqq \varphi_s \leqq 1$ be a smooth bumpfunction on $\mathbf{R}$ such that

$$\varphi_s(\beta) = 1 \qquad \text{if } |\beta| < 2^{-(s^\rho)},$$

$$\varphi_s(\beta) = 0 \qquad \text{if } |\beta| > 2.2^{-(s^\rho)}.$$

Define also

$$\Gamma_s = \{(a,q) \mid 1 \leqq a < q,\ (a,q) = 1 \text{ and } s \leqq q \leqq 2s\}.$$

Let

$$\hat{L}_n(\alpha) = v_n(\alpha) + \sum_{s\,\text{diadic}} \sum_{(a,q)\in\Gamma_s} S(q,a)v_n(\alpha - a/q)\varphi_s(\alpha - a/q). \tag{4.6}$$

LEMMA 4.6. $\| \hat{K}_n - \hat{L}_n \|_\infty < C \log n)^{-1/2\rho}$.

PROOF. By definition of φ_s, it follows that for each generation Γ_s, the functions $\varphi_s(\alpha - a/q)$ are disjointly supported. Moreover, by (5.2) and the fact that from van der Corput's lemma

$$|v_n(\beta)| < Cn^{-1}|\beta|^{-1/2}$$

for $\Gamma \subset \Gamma_s$

$$\begin{aligned} &\sum_{(a,q)\in\Gamma} |S(q,a)|\,\varphi_s(\alpha - a/q)\,|v_n(\alpha - a/q)| \\ &\leqq Cs^{-1/2} \sup_{(a,q)\in\Gamma} [1 + n|\alpha - a/q|^{1/2}]^{-1}. \end{aligned} \tag{4.7}$$

Assume first $\alpha \in \mathcal{M}(q,a)$, $s \leqq q \leqq 2s$. Then $\hat{K}_n(\alpha)$ is given by (4.4). We have $|\alpha - a'/q'| > 1/n$ if $q' < n^{1/2}$, $(a,q) \neq (a',q')$. Hence, by (4.7)

$$\hat{L}_n(\alpha) = S(q,a)v_n(\alpha - a/q)\varphi_s(\alpha - a/q) + O(n^{-1/4}). \tag{4.8}$$

If $s^\rho < \log n$, then $\varphi_s(\alpha - a/q) = 1$ and $|\hat{K}_n(\alpha) - \hat{L}_n(\alpha)| < Cn^{-1/4}$. If $s^\rho \geqq \log n$, then

$$|S(q,a)| < C(\log n)^{-1/2\rho}$$

and thus

$$|\hat{K}_n(\alpha) - \hat{L}_n(\alpha)| \leqq |\hat{K}_n(\alpha)| + |\hat{L}_n(\alpha)| < C(\log n)^{-1/2\rho}.$$

Next assume α is not in a major arc, thus $|\alpha| > n^{-2+\upsilon}$ and

$$|\alpha - a/q| < n^{-2+\upsilon} \Rightarrow q \geqq n^\upsilon.$$

By Dirichlet's principle, given α, we may find $a \leqq q$, $(a,q) = 1$ satisfying $q < n^{2-\upsilon}$, $|\alpha - a/q| \leqq q^{-1}n^{-2+\upsilon} < q^{-2}$. Thus, by assumption, $q > n^\upsilon$.

From (4.1), it follows that $|\hat{K}_n(\alpha)| < Cn^{-\upsilon/3}$. On the other hand, again by (4.7)

$$|\hat{L}_n(\alpha)| \leqq \sum_{\substack{s < n^\upsilon \\ s \text{ diadic}}} Cs^{-1/2}n^{-\upsilon/2} + \sum_{\substack{s > n^\upsilon \\ s \text{ diadic}}} s^{-1/2} = O(n^{-\upsilon/2})$$

completing the proof.

5. Proof of Theorem 1

Coming back to the discussion in Section 3, it follows from (4.7) that (3.1) holds provided $\rho < 1$. Thus we are reduced to evaluating in L^2-norm

$$\sup_{n\in Z} |f * L_n|$$

(5.1)
$$\leqq \sup_{n\in Z} \left| \int \hat{f}(\alpha) v_n(\alpha) e^{2\pi i\alpha x} d\alpha \right|$$

(5.2)
$$+ \sum_{s \text{ diadic}} \sup_{n\in Z} \left| \sum_{(a,q)\in\Gamma_s} S(q,a) \int \hat{f}(\alpha) v_n(\alpha - a/q) \varphi_s(\alpha - a/q) e^{2\pi i\alpha x} d\alpha \right|$$

Since the argument to bound (5.1) is contained in the argument to estimate the s-term in (5.2), we only consider the sum (5.2). Denote B_s the best constant satisfying an inequality

$$\left\| \sup_n |\mathscr{F}_{s,n}[g]| \right\|_2 \leqq B_s \| g \|_2 \tag{5.3}$$

where

$$\mathscr{F}_{s,n}[g](x)=\sum_{(a,q)\in\Gamma_s}\int \hat{g}(\alpha)v_n(\alpha-a/q)\varphi_s(\alpha-a/q)e^{2\pi i\alpha x}d\alpha.$$

Clearly

$$\sum_{(a,q)\in\Gamma_s} S(q,a)\int \hat{f}(\alpha)v_n(\alpha-a/q)\varphi_s(\alpha-a/q)e^{2\pi i\alpha x}d\alpha=\mathscr{F}_{s,n}[g_s]$$

defining

$$\hat{g}_s=\sum_{(a,q)\in\Gamma_s} S(q,a)\hat{f}\cdot\chi_{I_{a/q}}$$

where $I_{a/q}$ is a $2.2^{-(s^\rho)}$ neighborhood of a/q. Thus by (4.2) and (5.3)

$$\left\|\sup_n|\mathscr{F}_{s,n}[g_s]|\right\|_2\leqq B_s\|g_s\|_2\leqq CB_s s^{-1/2}\|f\|_2$$

and to control the sum in (5.2) it suffices to let B_s fulfil the condition

$$\sum B_s s^{-1/2}<\infty. \tag{5.4}$$

The remainder of this section deals with (5.3). Fix $\delta<\rho$ and consider a further splitting of Γ_s in $s^{1-\delta}$ sets Λ each of which only involves s^δ values of q, which consequently have common multiple $Q=Q_\Lambda$ satisfying

$$\log Q\leqq(\log s)s^\delta<s^\rho. \tag{5.5}$$

We prove that for each such set Λ

$$\left\|\sup_n\left|\sum_{(a,q)\in\Lambda}\int \hat{g}(\alpha)v_n(\alpha-a/q)\varphi(\alpha-a/q)e^{2\pi i\alpha x}d\alpha\right|\right\|_2\leqq C\|g\|_2 \tag{5.6}$$

where $\varphi=\varphi_s$ is supported by a τ-neighborhood of 0, $\tau Q\ll 1$ (from the definition of φ_s). In (5.6), C is a fixed constant, implying, by the triangle inequality,

$$B_s\leqq C\cdot s^{1-\delta}$$

and thus (5.4), taking $\frac{1}{2}<\delta<\rho<1$.

To obtain (5.6), the general estimate of Section 3 is used. Rewrite

$$v_n(\beta)=\int_0^1 e^{2\pi i n^2x^2\beta}dx=\frac{1}{2}\int_0^1 e^{2\pi i n^2\beta y}\frac{1}{\sqrt{y}}dy=\hat{k}(-n^2\beta)$$

where $k(y)=\frac{1}{2}y^{-1/2}\chi_{[0,1]}$ is in $L^1(\mathbf{R})$ (= quadratic density). Since k is decreasing, we may take $C(k)$ to be a constant in (3.2) and (3.5) completes the proof, since (3.3) is verified. The left member of (5.6) is indeed bounded by

$$\left\| \sup_{t>0} \left| \sum_{(a,q)\in\Lambda} \int \hat{k}(t(a/q-\alpha))\hat{g}(\alpha)e^{2\pi i\alpha x}\varphi(\alpha-a/q)d\alpha \right| \right\|_2 .$$

6. Proof of Theorem 2

The problem in the case of higher powers $\{n^t\}$, $t>2$ comes from worse bounds on the corresponding Weyl sums

$$S(q,a)=\frac{1}{q}\sum_{r=0}^{q-1} e^{2\pi i(a/q)r^t}, \qquad (a,q)=1. \tag{6.1}$$

Recall first H. Weyl's inequality.

LEMMA 6.2. *Let* $(a,q)=1$, $|\alpha-a/q|<1/q^2$ *and* $f(\alpha)=\sum_{m=0}^{n} e^{2\pi i m^t\alpha}$. *Then*

$$|f(\alpha)| \leqq n^{1+\varepsilon}[q^{-1}+n^{-1}+qn^{-t}]^{1/K} \quad \textit{where } K=2^{t-1}.$$

Let again $\upsilon=\frac{1}{100}$ and consider the same system of major arcs

$$\mathcal{M}_0=\{\alpha\in\mathbf{T} \mid |\alpha|<n^{-t+\upsilon}\}.$$

For $(a,q)=1$, $1\leqq a<q<n^\upsilon$

$$\mathcal{M}(q,a)=\{\alpha\in\mathbf{T} \mid |\alpha-a/q|<n^{-t+\upsilon}\}.$$

We then have by [Vaug] (Lemma 2.7)

LEMMA 6.3. *Let* $\hat{K}_n(\alpha)=(1/n)\sum_{m=0}^{n} e^{2\pi i m^t\alpha}$ *and* $\alpha\in\mathcal{M}_0$ *or* $\alpha\in\mathcal{M}(q,a)$. *Then*

$$\hat{K}_n(\alpha)=S(q,s)v_n(\alpha-a/q)+O(n^{-1/2}) \tag{6.4}$$

where now $S(q,a)$ *is given by* (6.1) *and*

$$v_n(\beta)=\int_0^1 e^{2\pi i n^t x^t\beta}dx.$$

If α *is not in a major arc, then by Dirichlet's principle and Lemma* 6.2,

$$|\hat{K}_n(\alpha)| \leqq n^{-v/K+\varepsilon}. \tag{6.5}$$

In the next lemma, information about the $S(q,a)$ is summarized.

LEMMA.

(6.6) $$|S(q,a)| \leqq Cq^{-1/K+\varepsilon}.$$

(6.7) *If q is prime and $(a,q)=1$, then* $|S(q,a)| < Cq^{-1/2}$.

(6.8) *If q is prime, $(a,q)=1$ and $k>1$, then*

$$|S(q^k,a)| < Cq^{-1}.$$

(6.9) *Let p be a prime $(p,q)=1$ and $(a,pq)=1$. Then*

$$|S(pq,a)| \leqq Cp^{-1/2}q^{-1/K+\varepsilon},$$

$$|S(p^kq,a)| \leqq Cp^{-1}q^{-1/K+\varepsilon} \quad \textit{for } k>1.$$

PROOF. (6.6) follows from (6.2) taking $\alpha = a/q$, $n=q$. (6.7) is a particular case of an inequality due to A. Weil (see [L–N] for an elementary exposition).

To prove (6.8), we may assume $(q,t)=1$. Writing for $0 \leqq m < q^k$,

$$m = y + zq^{k-1}; \qquad 0 \leqq y < q^{k-1}, \quad 0 \leqq z < q,$$

we have

$$\sum_{0\leqq m<q^k} e^{2\pi i m^t a q^{-k}} = \sum_{0\leqq y<q^{k-1}} e^{2\pi i y^t a q^{-k}} \left(\sum_{0\leqq z<q} e^{2\pi i y^{t-1} t z a q^{-1}} \right)$$

$$= q \sum_{\substack{0\leqq y<q^{k-1} \\ q\mid y}} e^{2\pi i y^t a q^{-k}}$$

implying (6.8).

To prove (6.9), observe that the residue classes mod p^kq have a unique representation

$$m = xp^k + yq; \qquad 0 \leqq x < q, \quad 0 \leqq y < p^k.$$

Hence

$$\sum_{0\leqq m<p^kq} e^{2\pi i m^t a/p^k q} = \left(\sum_{0\leqq x<q} e^{2\pi i x^t a' q^{-1}} \right) \left(\sum_{0\leqq y<p^k} e^{2\pi i y^t a'' p^{-k}} \right)$$

where $a' = ap^{k(t-1)}$, $a'' = aq^{t-1}$, hence $(a',q) = 1 = (a'',p)$. Consequently

$$|S(p^kq,a)| = |S(q,a')| \cdot |S(p^k,a'')|$$

and (6.9) follows immediately from (6.6) and (6.7), (6.8).

LEMMA 6.10. *Assume* $(a, q) = 1$, *q has a prime factor at least equal to s and* $q > s^{1+\delta}$. *Then*

$$|S(q, a)| < Cs^{-1/2-\delta'}, \quad \delta' < \delta/K.$$

PROOF. Write $q = p^k q'$ with $p \geqq s$, p prime and $(p, q') = 1$. Apply (6.9). If $k > 1$, $|S(q, a)| \leqq s^{-1}$. Otherwise

$$|S(q, a)| < p^{-1/2}(q/p)^{-1/K+\varepsilon} < s^{-1/2+1/K} s^{-(1+\delta)(1/K-\varepsilon)}. \qquad \blacksquare$$

Our next purpose is to write an appropriate approximation $\hat{L}_n(\alpha)$ for $\hat{K}_n(\alpha)$, such that

$$\| \hat{L}_n - \hat{K}_n \|_\infty < (\log n)^{-\sigma} \qquad (\sigma > \tfrac{1}{2})$$

and $\| \sup_n |f * L_n| \|_2$ may be estimated similarly as in the quadratic case.

Fix $0 < \rho < 1 < \kappa$ and $\delta > 0$. Let for diadic s, $0 \leqq \varphi_s \leqq 1$ and $0 \leqq \psi_s \leqq 1$ be smooth functions satisfying

$$\varphi_s(\beta) = 1 \quad \text{if } |\beta| < 2^{-(s^\rho)} \quad \text{and} \quad \varphi_s(\beta) = 0 \quad \text{for } |\beta| > 2.2^{-(s^\rho)},$$

$$\psi_s(\beta) = 1 \quad \text{if } |\beta| < 2^{-(s^\kappa)} \quad \text{and} \quad \psi_s(\beta) = 0 \quad \text{for } |\beta| > 2.2^{-(s^\kappa)}.$$

Define

$$Q_s = [s!]^{C(t)[\log s]}$$

where $C(t)$ is a large integer depending on t.

Let further

$$\bar{\Gamma}_s = \{(a, q) \mid (a, q) = 1 \text{ and } q \mid Q_s\},$$

$$\Gamma_s = \bar{\Gamma}_s \setminus \bar{\Gamma}_{s-1},$$

and define

$$\hat{L}_n(\alpha) = v_n(\alpha) + \sum_{s \text{ diadic}} \sum_{(a,q) \in \Gamma_s} S(q, a) v_n(\alpha - a/q) \psi_s(\alpha - a/q) \tag{6.11}$$

$$+ \sum_{s \text{ diadic}} \sum_{\substack{(a,q) \in \Gamma_s \\ q \leqq s^{1+\delta}}} S(q, a) v_n(\alpha - a/q)(\varphi_s - \psi_s)(\alpha - a/q). \tag{6.12}$$

LEMMA 6.13. $|\hat{K}_n(\alpha) - \hat{L}_n(\alpha)| \leqq (\log n)^{-\sigma}$ *for some* $\sigma > \frac{1}{2}$.

Observe that for each s in (6.11), $q \mid Q_s$ if $(a, q) \in \Gamma_s$ where

$$\log Q_s \sim s(\log s)^2 < s^\kappa \qquad (\kappa > 1).$$

The bumpfunction ψ_s is thus supported by a τ-neighborhood of 0 where $\tau \cdot Q_s \ll 1$. Therefore, (6.11) gives a maximal function estimated as

$$\sum_{s \text{ diadic}} \sup_{(a,q)\in\Gamma_s} |S(q,a)| \tag{6.14}$$

using again the rsult of Section 3, where now

$$k(y) = \frac{1}{ty^{1-1/t}} \chi_{[0,1]}.$$

Notice that, by definition of Q_s, if $(a,q) \notin \bar{\Gamma}_{s-1}$, necessarily $q \geqq s$ and $|S(q,a)| < s^{-1/2K}$. Hence (6.14) is bounded. To estimate the contribution of the terms in (6.12), split

$$\Gamma_s \cap \{(a,q) \mid q \leqq s^{1+\delta}\}$$

is $s^{\delta+(1-\rho')}$ sets Λ, each of which only involves $s^{\rho'}$ values of q. Taking $\rho' < \rho$, these q's will have a common multiple Q satisfying

$$\log Q \leqq s^{\rho'} \log s < s^{\rho} \Rightarrow \tau Q \ll 1$$

letting $\tau = 2.2^{-(s^\rho)}$. since φ_s is supported by a τ-neighborhood of 0, the estimate in Section 3 appies to each set Λ. The contribution of (6.12) is evaluated as

$$\sum_{s \text{ diadic}} s^{-1/K+\varepsilon} s^{\delta+(1-\rho')} < \infty$$

provide

$$1/K > \delta + (1-\rho') + \varepsilon, \text{ leading to the condition } \delta + (1-\rho) < 1/K. \tag{6.15}$$

Proof of (6.13). Observe that again by van der Corput's lemma

$$|v_n(\beta)| = \left| \int_0^1 e^{2\pi i n^t \beta x^t} dx \right| \leqq C n^{-1} |\beta|^{-1/t}$$

and that for fixed s

$$\{\psi_s(\alpha - a/q) \mid (a,q) \in \Gamma_s\}; \quad \{\varphi_s(\alpha - a/q) \mid q \leqq s^{1+\delta}\}$$

are disjointly supported functions. Therefore, if $\Gamma \subset \Gamma_s$

$$\begin{aligned} &\left| \sum_{(a,q)\in\Gamma} S(q,a) v_n(\alpha - a/q) \psi_s(\alpha - a/q) \right| \\ &\leqq C \sup_{(a,q)\in\Gamma} \{ |S(q,a)| \cdot [1 + n|\alpha - a/q|^{1/t}]^{-1} \}, \end{aligned} \tag{6.16}$$

$$\left| \sum_{\substack{(a,q)\in\Gamma \\ q \leqq s^{1+\delta}}} S(q,a) v_n(\alpha - a/q) \varphi_s(\alpha - a/q) \right|$$

(6.17)
$$\leqq C \sup_{(a,q)\in\Gamma} \{ |S(q,a)| \cdot [1 + n\,|\alpha - a/q|^{1/t}]^{-1} \}.$$

Assume first α belongs to a major arc $\mathcal{M}_0$ or $\mathcal{M}(q,a)$, $q < n^{\upsilon}$. Then $\hat{K}_n(\alpha)$ is given by (6.4)

$$\hat{K}_n(\alpha) = S(q,a) v_n(\alpha - a/q) + O(n^{-1/2}).$$

Assume $(a,q) \in \Gamma_s$. For each s', let $\tilde{\Gamma}_{s'} = \Gamma_{s'} \setminus \{(a,q)\}$. Then

$$\begin{aligned} \hat{L}_n(\alpha) &= S(q,a) v_n(\alpha - a/q) \psi_s(\alpha - a/q) \\ &\quad + S(q,a) v_n(\alpha - a/q)(\varphi_s - \psi_s)(\alpha - a/q) \quad (\text{if } q < s^{1+\delta}) \\ &\quad + \text{error term } E \end{aligned}$$

where by (6.16), (6.17)

(6.18)
$$E \leqq \sum_{s' \text{ diadic}} \sup_{(q',a')\in\tilde{\Gamma}_{s'}} \{ |S(q',a')| \cdot [1 + n\,|\alpha - a'/q'|^{1/t}]^{-1} \}.$$

Notice that if $(a,q) \neq (a',q')$, $q' < n$, then

$$\left| \frac{a}{q} - \frac{a'}{q'} \right| > \frac{1}{n^{1+\nu}} \Rightarrow \left| \alpha - \frac{a'}{q'} \right| > 2n^{-1-\nu}.$$

If $q' > n$, then $|S(q',a')| < n^{-1/2K}$. Also, if $(q',a') \in \Gamma_{s'}$, necessarily $q' > s'$, hence

$$|S(q',a')| < (s')^{-1/2K}.$$

Thus (6.18) is bounded by

$$E \leqq \sum_{s' \text{ diadic}} (s')^{-1/2K} \wedge n^{-1/2K} < C n^{-1/3K}.$$

Assume $q < s^{1+\delta}$. Then

$$\hat{L}_n(\alpha) = S(q,a) v_n(\alpha - a/q) \varphi_s(\alpha - a/q) + O(n^{-1/3K}).$$

If $\varphi_s(\alpha - a/q) \neq 1$, then

$$n^{\upsilon - t} > |\alpha - a/q| > 2^{-(s^{\rho})} \Rightarrow s > (\log n)^{1/\rho}.$$

By definition of Q_s and the fact that $(a, q) \notin \bar{\Gamma}_{s-1}$, q has a prime factor at least s and consequently, by (6.9),

$$|S(q, a)| < Cs^{-1/2} < C(\log n)^{-\sigma}, \quad \sigma = 1/2\rho > \tfrac{1}{2}.$$

Hence

$$|\hat{K}_n(\alpha) - \hat{L}_n(\alpha)| \lesssim |S(q, a)| < C(\log n)^{-\sigma}.$$

Assume $q > s^{1+\delta}$. It follows from (6.10) that $|S(q, a)| < s^{-1/2-\delta'}$ provided q has a prime factor $\geqq s$. Otherwise, by definition of Q_{s-1}, $\log q > C(t) \cdot \log s$. In the last case, using (6.6), for appropriate $C(t)$, $|S(q, a)| < s^{-1}$. We have

$$\hat{K}_n(\alpha) = S(q, a) v_n(\alpha - a/q) + O(n^{-1/2}),$$

$$\hat{L}_n(\alpha) = S(q, a) v_n(\alpha - a/q) \psi_s(\alpha - a/q) + O(n^{-1/3K}).$$

If $\psi_s(\alpha - a/q) \neq 1$, then

$$n^{-t+v} > |\alpha - a/q| > 2^{-(s^{\kappa})} \Rightarrow s > (\log n)^{1/\kappa}$$

and consequently

$$|\hat{K}_n(\alpha) - \hat{L}_n(\alpha)| \lesssim |S(q, a)| < (\log n)^{-(1/\kappa)(1/2+\delta')} < (\log n)^{-\sigma}, \quad \sigma > \tfrac{1}{2}$$

provided

$$1 + 2\delta' > \kappa.$$

This establishes (6.13) for α in a major arc.

Thus it remains to consider the case $\alpha \notin \mathcal{M}_0$ and $\alpha \notin \mathcal{M}(q, a)$ for $q < n^v$. By (6.5), $|\hat{K}_n(\alpha)| < n^{-v/K+\varepsilon}$. Estimate $\hat{L}_n(\alpha)$ by (6.16), (6.17) as

$$|\hat{L}_n(\alpha)| \leqq n^{-1}|\alpha|^{-1/t} + \sum_{s \text{ diadic}} \sup_{(a,q) \in \Gamma_s} \{|S(q, a)| \cdot [1 + n|\alpha - a/q|^{1/t}]^{-1}\}.$$

By hypothesis, $|\alpha| > n^{-t+v}$ and $|\alpha - a/q| > n^{-t+v}$ for $q < n^v$. If $q \geqq n^v$, $|S(q, a)| < n^{-v/2K}$. Also, again $|S(q, a)| < s^{-1/2K}$ for $(a, q) \in \Gamma_s$, since $q \geqq s$. Hence

$$|\hat{L}_n(\alpha)| \leqq n^{-v/t} + \sum_{s \text{ diadic}} (s^{-1/2K} \wedge n^{-v/2K}) < Cn^{-v/3K}.$$

This completes the proof of (6.13) and hence of Theorem 2.

7. Proof of Theorem 5

Suppose $\| f \|_2 \leqq 1$, $\| f \|_\infty \leqq 1$ and $\mathscr{A}_n f = (1/n) \sum_{m=1}^n T^{(m^t)} f$ does not converge a.s. Then there is $\tau > 0$ and an increasing sequence of integers $n_0 < n_1 < \cdots < n_{k-1} < n_k < \cdots$, $n_k > 2n_{k-1}$, such that for each k

$$\mu\left[\max_{n_{k-1}<n<n_k} |\mathscr{A}_n f - \mathscr{A}_{n_{k-1}} f| > \tau\right] > \tau. \tag{7.1}$$

Take $\varepsilon = \tau/10$ and define $S = \{[1+\varepsilon)^r];\ r = 1, 2, \ldots\}$. Since $\| f \|_\infty \leqq 1$, for $n' \equiv (1+\varepsilon)^r \leqq n \leqq (1+\varepsilon)^{r+1}$

$$|\mathscr{A}_n f - \mathscr{A}_{n'} f| \leqq 4\varepsilon.$$

Thus, letting $S_k = S \cap [n_{k-1}, n_k]$,

$$\max_{n_{k-1}<n<n_k} |\mathscr{A}_n f - \mathscr{A}_{n_{k-1}} f| \leqq \max_{n\in S_k} |\mathscr{A}_n f - \mathscr{A}_{n_{k-1}} f| + 4\varepsilon,$$

$$\mu\left[\max_{n\in S_k} |\mathscr{A}_n f - \mathscr{A}_{n_{k-1}} f| > \tau/2\right] > \tau. \tag{7.2}$$

Our purpose is to show that taking $K = K(\tau)$ sufficiently large, (7.2) cannot hold for $k = 1, \ldots, K$. This is a statement of "finite nature" and, as for the maximal operator, the general case is equivalent to the special case of the shift on Z. The argument is similar to the one appearing in Section 2. Thus for fixed $x \in \Omega$, define φ on Z by

$$\varphi(j) = f(T^j x) \qquad \text{for } |j| \leqq J$$
$$= 0 \qquad \text{for } |j| > J$$

when J is taken large enough (depending on n_K). Assume the following statement holds:

Lemma 7.3. *If $\| \varphi \|_\infty \leqq 1$ is a finitely supported function on Z, then*

$$\sum_{1\leqq k\leqq K} \left\| \max_{n\in S_k} |\mathscr{A}_n \varphi - \mathscr{A}_{n_{k-1}} \varphi| \right\|_2^2 \leqq \theta(K) K \| \varphi \|_2^2$$

when $\theta(K) \to 0$ for $K \to \infty$.

We thus obtain by definition of φ, letting $R = n_K^t$,

$$\sum_{k=1}^K \sum_{|j|<J-R} \left[\max_{n\in S_k} |(\mathscr{A}_n f - \mathscr{A}_{n_{k-1}} f)(T^j x)|\right]^2 \leqq \theta K \sum_{|j|\leqq J} |f(T^j x)|^2.$$

Integrating in $x \in \Omega$, since T is measure preserving

$$(J-R)\sum_{k=1}^{K}\left\|\max_{n\in S_k}|\mathscr{A}_n f-\mathscr{A}_{n_{k-1}}f|\right\|_2^2 \leqq \theta KJ\,\|f\|_2^2$$

and (7.2) implies

$$\frac{J-R}{J}\frac{\tau^3}{4}<\theta(K)\,\|f\|_2^2 \quad \text{hence} \quad \tau^3<4\theta(K),$$

a contradiction for $K \to \infty$.

In the case of the shift (Z, S), we have $\mathscr{A}_n\varphi = \varphi * K_n$ when $K_n = (1/n)\sum_{m=1}^{n}\delta_{\{m^t\}}$. Again, with L_n defined by (6.11), (6.12) and $n_k = (1+\varepsilon)^{r_k}$

$$\begin{aligned}
&\left\|\max_{n\in S_k}|\mathscr{A}_n\varphi-\mathscr{A}_{n_{k-1}}\varphi|\right\|_2\\
&\leqq \left\|\max_{n\in S_k}|(\varphi * L_n)-(\varphi * L_{n_{k-1}})|\right\|_2+\left(\sum_{n\in S_k}\|\hat{K}_n-\hat{L}_n\|_\infty^2\right)^{1/2}\\
&\leqq \left\|\max_{n\in S_k}|\varphi *(L_n-L_{n_{k-1}})|\right\|_2+\varepsilon^{-\sigma}\left(\sum_{r_{k-1}<r<r_k} r^{-2\sigma}\right)^{1/2}\|\varphi\|_2
\end{aligned}$$

and the left member of (7.3) is bounded by

$$c\sum_{1\leqq k\leqq K}\left\|\max_{n\in S_k}|\varphi *(L_n-L_{n_{k-1}})|\right\|_2^2+c(\tau)\,\|\varphi\|_2^2. \tag{7.4}$$

Coming back to the definition of $\hat{L}_n(\alpha)$ given by (6.11), (6.12), we may write

$$\hat{L}_n(\alpha)=v_n(\alpha)+\sum_{S\text{ diadic}}\ \sum_{(a,q)\in\Lambda_S} S(q,a)v_n(\alpha-a/q)\eta_S(\alpha-a/q) \tag{7.5}$$

where

$$v_n(\beta)=\int_0^1 e^{2\pi i n^t x^t\beta}dx,$$

$(a,q)\in\Lambda_S \Rightarrow q \mid Q_S$ and $0 \leqq \eta_S \leqq 1$ is a smooth bumpfunction. As shown in the previous section

$$\left\|\sup_n\left|\int\hat{\varphi}(\alpha)\left[\sum_{(a,q)\in\Lambda_S}S(q,a)v_n(\alpha-a/q)\eta_S(\alpha-a/q)\right]e^{2\pi i\alpha x}d\alpha\right|\right\|_2<cs^{-\kappa}$$

(7.6)

for some $\kappa > 0$. Thus the larger values of s will have a small contribution in (7.4). More precisely, fix s_0 and estimate, by (7.6),

$$(7.7)\quad \left\| \max_{n\in S_k} |\varphi * (L_n - L_{n_{k-1}})| \right\|_2 \leqq \left\| \max_{n\in S_k} |\varphi * (M_n - M_{n_{k-1}})| \right\|_2 + c \sum_{\substack{s\geqq s_0\\ s \text{ diadic}}} s^{-\kappa}$$

letting

$$(7.8)\qquad \hat{M}_n(\alpha) = v_n(\alpha) + \sum_{s\leqq s_0} \sum_{(a,q)\in\Lambda_S} S(q,a) v_n(\alpha - a/q)\eta_S(\alpha - a/q).$$

Estimate the sum in (7.4) as

$$Ks_0^{-\kappa} + \sum_{1\leqq k\leqq K} \left\| \max_{n\in S_k} \left| \int \hat{\varphi}(\alpha)(v_n - v_{n_{k-1}})(\alpha) e^{2\pi i x\alpha} d\alpha \right| \right\|_2^2$$

$$(7.9)\qquad + |\bar{\Lambda}| \sum_{\substack{1\leqq s\leqq s_0\\ s\text{ diadic}}} \sum_{(a,q)\in\Lambda_S} \sum_{1\leqq k\leqq K}$$

$$\times \left\| \max_{n\in S_k} \left| \int \hat{\varphi}(\alpha)(v_n - v_{n_{k-1}})(\alpha - a/q)\eta_S(\alpha - a/q) e^{2\pi i x\alpha} d\alpha \right| \right\|_2^2$$

where $\bar{\Lambda} = \bigcup_{s\leqq s_0} \Lambda_S$.

This estimate follows from (7.7), (7.8), the triangle inequality and Cauchy–Schwarz.

Let $0 \leqq \eta \leqq 1$ be a smooth function vanishing outside a neighborhood of 0. Assume there is a constant B, independent of K, satisfying the inequality

$$(7.10)\quad \sum_{1\leqq k\leqq K} \left\| \max_{n\in S_k} \left| \int \hat{g}(\alpha)(v_n - v_{n_{k-1}})(\alpha)\eta(\alpha) e^{2\pi i x\alpha} d\alpha \right| \right\|_2^2 \leqq B \, \| g \|_2^2.$$

Then (7.9) is bounded by

$$Ks_0^{-\kappa} + c(s_0)B$$

and consequently we may take in (7.3)

$$\theta = s_0^{-\kappa} + \frac{B}{K} c(s_0).$$

Thus, for an appropriate choice of $s_0 = s_0(K)$, it follows that $\theta(K) \to 0$ when $K \to \infty$. It remains to prove (7.10). Recall that

$$v_n(\alpha) = \hat{k}(-n^t\alpha) \quad \text{with } k(y) = \tfrac{1}{t} y^{1/t-1} \chi_{[0,1]}$$

and the inequality

$$(7.11)\qquad \Big\| \sup_{\lambda>0} |h * k_\lambda| \Big\|_{L^2(R)} = c \, \| h \|_{L^2(R)}, \qquad k_\lambda(y) = \lambda^{-1} k(\lambda^{-1} y)$$

used earlier in this paper.

A similar reasoning exploited in Section 3 permits one to deduce (7.10) from the corresponding inequality on R. Thus one replaces $x \in \mathbf{Z}$ by $x + u$, $u \in [0, 1]$, and uses the fact that $|e^{2\pi i \alpha u} - 1| < \frac{1}{10}$ say, for α in the support of η.

Consider for each k, smooth functions $0 \leqq \omega_k, \psi_k^+, \psi_k^- \leqq 1$ satisfying

$$\omega_k(\alpha) = 1 \text{ if } n_k^{-t} \leqq |\alpha| \leqq n_{k-1}^{-t} \text{ and } \omega_k(\alpha) = 0 \text{ if } |\alpha| < \tfrac{1}{2} n_k^{-t} \text{or } |\alpha| > 2 n_{k-1}^{-t},$$

$$\psi_k^+(\alpha) = 1 - \omega_k(\alpha) \text{ if } |\alpha| > \tfrac{1}{2} n_{k-1}^{-t} \text{ and } \psi_k^+(\alpha) = 0 \text{ otherwise,}$$

$$\psi_k^-(\alpha) = 1 - \omega_k(\alpha) \text{ if } |\alpha| < 2 n_{k-1}^{-t} \text{ and } \psi_k^-(\alpha) = 0 \text{ otherwise.}$$

Estimate (7.10) as

$$(7.12)\qquad \sum_{1 \leqq k \leqq K} \Big\| \max_n \Big| \int \hat{g}(\alpha) \omega_k(\alpha) v_n(\alpha) \eta(\alpha) e^{2\pi i x \alpha} d\alpha \Big\|_2^2$$

$$(7.13)\qquad + \sum_{1 \leqq k \leqq K} \sum_{n \in S_k} \Big\| \int \hat{g}(\alpha) \psi_k^+(\alpha) v_n(\alpha) \eta(\alpha) e^{2\pi i x \alpha} d\alpha \Big\|_2^2$$

$$(7.14)\qquad + \sum_{1 \leqq k \leqq K} \sum_{n \in S_k} \Big\| \int \hat{g}(\alpha) \psi_k^-(\alpha) (v_n - v_{n_{k-1}})(\alpha) \eta(\alpha) e^{2\pi i x \alpha} d\alpha \Big\|_2^2$$

where $\| \quad \|_2$ refers to the $L^2(R)$-norm.

By (7.11), (7.12) is bounded by ($^\vee$ referring to the inverse Fourier transform)

$$\sum_k \Big\| \sup_{\lambda>0} |g * \check{\eta} * \check{\omega}_k * k_\lambda| \Big\|_2^2 \leqq c \sum_k \| g * \check{\omega}_k \|_2^2$$

$$= c \int |\hat{g}(\alpha)|^2 \Big(\sum_k |\omega_k(\alpha)|^2 \Big) d\alpha$$

$$< c \, \| g \|_2^2$$

by definition of ω_k.

Estimate (7.13) by Parseval and using the fact that

$$|v_n(\alpha)| \leqq c n^{-1} |\alpha|^{-1/t}.$$

Thus, letting $\beta = |\alpha|^{1/t}$

$$\leqq c \| g \|_2^2 \sup_{\beta>0} \left[\sum_k \sum_{n\in S_k} \chi_{[(1/2)n_{k-1}^{-1},\infty[}(\beta) n^{-1}\beta^{-1} \right]$$

$$\leqq c \| g \|_2^2 \sup_{\beta>0} \left[\sum_{n_{k-1}^{-1}\leqq 2\beta} \varepsilon^{-1} n_{k-1}^{-1}\beta^{-1} \right] \tag{7.13}$$

$$\leqq c(\varepsilon) \| g \|_2^2$$

using the definition of $S_k = [n_{k-1}, n_k] \cap \{[(1+\varepsilon)^r]\}$ in the second inequality.

To estimate (7.14), write for $n \in S_k$

$$|(\nu_n - \nu_{n_{k-1}})(\alpha)| \leqq |1 - \nu_n(\alpha)| + |1 - \nu_{n_{k-1}}(\alpha)| \leqq c\,|\alpha|\,n^t.$$

Hence, again by Parseval

$$\leqq c \| g \|_2^2 \sup_{\beta>0} \left[\sum_k \sum_{n\in S_k} \chi_{[0,2n_k^{-1}]}(\beta) n\beta \right] \leqq c\varepsilon^{-1} \| g \|_2^2 \tag{7.14}$$

which completes the proof of Lemma 7.3 and hence of Theorem 5.

8. Appendix 1: Pointwise ergodic theorem for random sets

There is a simple way of generating thin sequences of integers satisfying a pointwise ergodic theorem on L^p, $p > 1$, by choosing integers at random with appropriate density. Recall that a sequence $S \subset Z_+$ is ergodic provided

$$\frac{1}{|S \cap [0, N]|} \sum_{\substack{k\in S \\ k\leqq N}} e^{2\pi i k x} \xrightarrow{N\to\infty} 0 \quad \text{for } x \in \Pi \setminus \{0\}. \tag{8.1}$$

An ergodic sequence satisfies the mean ergodic theorem

Proposition 8.2. *Let $(\sigma_n)_{n=1,2,\dots}$ be a decreasing sequence of positive numbers and $S \subset \mathbf{Z}_+$ the random set obtained by including each integer $n \in \mathbf{Z}_+$ in the set S with probability σ_n.*

(i) *If $\lim_{n\to\infty} n\sigma_n = \infty$, then S is almost surely ergodic.*

(ii) *If $\sigma_n = n^{-1}(\log\log n)^B$, $B > (p-1)^{-1}$ $(1 < p \leqq 2)$, then almost surely S satisfies the L^p-maximal and pointwise ergodic theorem.*

The main ingredient of the proof is the following fact.

Lemma 8.3. *Fix $N \geqq 1$ and let $S \subset [0, N]$ be the random subset corresponding to the sequence of probabilities $(\sigma_n)_{1\leqq n\leqq N}$. Then with probability at least $1 - 1/N^2$*

$$(8.4)\qquad \left[\frac{1}{|S|}\sum_{n\in S} z^n - \frac{1}{\sum_{1\leqq n\leqq N}\sigma_n}\sum_{n=1}^{N}\sigma_n z^n\right] \leqq c\left\{\log N \Big/ \sum_{1\leqq n\leqq N}\sigma_n\right\}^{1/2}$$

for all $z\in\mathbf{C}$, $|z|=1$.

PROOF. Denote $\{\xi_n\}_{1\leqq n\leqq N}$ independent (0,1)-valued selectors of mean σ_n. Thus $S=S_\omega=\{1\leqq n\leqq N \mid \xi_n(\omega)=1\}$. For $q\geqq 1$ denote

$$I_q = \left\| \sum_{1\leqq n\leqq N}\xi_n\right\|_q .$$

Writing, based on Khintchine's inequality,

$$I_q \leqq \left(\sum_{1\leqq n\leqq N}\sigma_n\right) + \left\|\sum_{n\leqq N}(\xi_n-\sigma_n)\right\|_q \leqq \left(\sum_{n\leqq N}\sigma_n\right)$$

$$(8.6)\qquad +2\left\|\sum_{n\leqq N}\varepsilon_n(\xi_n-\sigma_n)\right\|_{L^q(d\omega\otimes d\varepsilon)}$$

$$\leqq \left(\sum_{n\leqq N}\sigma_n\right) + 2\sqrt{q}\,\|(\Sigma|\xi_n-\sigma_n|^2)^{1/2}\|_q \leqq \left(\sum_{n\leqq N}\sigma_n\right)+4\sqrt{q}I_q^{1/2}$$

it follows that

$$(8.5)\qquad I_q \leqq c\max\left(q,\sum_{n\leqq N}\sigma_n\right),$$

$$(8.6)\qquad \left\|\sum_{n\leqq N}a_n(\xi_n-\sigma_n)\right\|_q \leqq c\max\left\{q,\sqrt{q}\left(\sum_{n\leqq N}\sigma_n\right)^{1/2}\right\}\quad \text{if } |a_n|=1.$$

Writing

$$D(\omega,z)\equiv\left|\frac{1}{\sum_{n\leqq N}\xi_n}\left(\sum_{n\leqq N}\xi_n(\omega)z^n\right)-\frac{1}{\sum_{n\leqq N}\sigma_n}\left(\sum_{n\leqq N}\sigma_n z^n\right)\right|$$

$$\leqq \left|\frac{|\Sigma(\xi_n-\sigma_n)|}{\Sigma\sigma_n}+\frac{|\Sigma(\xi_n-\sigma_n)z^n|}{\Sigma\sigma_n}\right|$$

and using the fact that $\sup_{|z|=1}D(\omega,z)$ may be evaluated taking z in *a* net $\mathscr{E}$ of $2N$ points (by Bernstein's inequality), it follows from (8.6) applied with $q=\log N\ll\sum_{n\leqq N}\sigma_n$ that

$$
\begin{aligned}
\Big\| \sup_{|z|=1} D(\omega, z) \Big\|_{L^q(d\omega)} &\leqq c \, \Big\| \sup_{z \in \mathscr{E}} D(\omega, z) \Big\|_q \\
&\leqq 2c \sup_{z \in \mathscr{E}} \| D(\omega, z) \|_q \qquad (8.7) \\
&< c' \Big\{ \log N \Big/ \sum_{n \leqq N} \sigma_n \Big\}^{1/2}.
\end{aligned}
$$

Consequently, for an appropriate choice of the constant c in (8.4), (8.4) will hold with probability at least (by Techebychev's inequality and (8.7))

$$
1 - c^{-q} \left\{ \frac{\Sigma \sigma_n}{\log N} \right\}^{-q/2} \Big\| \sup_{|z|=1} D(\omega, z) \Big\|_q^q > 1 - \frac{1}{N^2}.
$$

Lemma 8.8. *Let S be the random set considered in Proposition 8.2. Then almost surely*

$$
\sup_N \left\{ \frac{\Sigma_{n \leqq N} \sigma_n}{\log N} \right\}^{1/2} \left\{ \sup_{|z|=1} \left| \frac{1}{|S_N|} \sum_{n \in S_N} z^n - \frac{1}{\Sigma_{n \leqq N} \sigma_n} \sum_{n \leqq N} \sigma_n z^n \right| \right\} < \infty
$$

where $S_N = S \cap [0, N]$.

Proof. It follows from Lemma 8.3 that S_N fulfils (8.4) with probability at least $1 - 1/N^2$. Lemma 8.8 is now straightforward.

Lemma 8.9. *Let S be the random set of Proposition 8.2 and $1 < p \leqq 2$. Then almost surely, for all N and some constant c, the following inequality holds for an arbitrary dynamical system (Ω, μ, T):*

$$
(8.9) \quad \left\| \frac{1}{|S_N|} \Big(\sum_{n \in S_N} T^n f \Big) - \frac{1}{\Sigma_{n \leqq N} \sigma_N} \Big(\sum_{n=1}^{N} \sigma_n T^n f \Big) \right\|_p \leqq c \left\{ \frac{\log N}{\Sigma_{n \leqq N} \sigma_n} \right\}^{1/p'} \| f \|_p .
$$

Proof. Obviously, for $p = 1$, (8.9) holds with $c = 2$. Hence, by interpolation, it suffices to consider the case $p = 2$. If c satisfies

$$
\sup_{|z|=1} \left| \frac{1}{|S_N|} \sum_{n \in S_N} z^n - \frac{1}{\Sigma_{n \leqq N} \sigma_n} \sum_{n \leqq N} \sigma_n z^n \right| < c \left(\frac{\log N}{\Sigma_{n \leqq N} \sigma_n} \right)^{1/2}
$$

von Neumann's theorem implies, for an arbitrary unitary transformation U on a Hilbert space H,

$$
\left\| \frac{1}{|S_N|} \sum_{n \in S_N} U^n - \frac{1}{\Sigma_{n \leqq N} \sigma_n} \sum_{n \leqq N} \sigma_n U^n \right\|_{B(H)} \leqq c \left(\frac{\log N}{\Sigma_{n \leqq N} \sigma_n} \right)^{1/2}.
$$

In particular, (8.9) holds for $p = 2$.

PROOF OF PROPOSITION 8.2. (i) If $\lim_{n\to\infty} \sigma_n n = \infty$, then $(\log N)/\Sigma_{n\leqq N}\sigma_n \to 0$ for $N\to\infty$.

Also, since $\sigma_n \searrow$ and $\Sigma_{n=1}^{\infty}\sigma_n = \infty$,

$$\frac{1}{\sum_{n\leqq N}\sigma_n}\sum_{n\leqq N}\sigma_n z^n \to 0 \quad \text{for } z \neq 1.$$

Hence, by Lemma 8.8,

$$\frac{1}{|S_N|}\sum_{n\in S_N} z^n \to 0 \quad \text{for } z \neq 1.$$

(ii) It follows from (8.9) that

$$\left\| \frac{1}{|S_n|}\left(\sum_{n\in S_N} T^n f\right) - \frac{1}{\Sigma_{n\leqq N}\sigma_n}\left(\sum_{n\leqq N}\sigma_n T^n f\right)\right\|_p \leqq c(\log\log N)^{-B/p'}\,\|f\|_p. \tag{8.10}$$

Again, since $\sigma_n \searrow$ and $\Sigma\sigma_n = \infty$, as a consequence of the usual pointwise ergodic theorem

$$\frac{1}{\sum_{n\leqq N}\sigma_n}\sum_{n\leqq N}\sigma_n T^n f \xrightarrow{N\to\infty} L(f) \quad \text{a.s.} \tag{8.11}$$

for $f\in L^p(\Omega,\mu)$, $p\geqq 1$. Here $L(f)$ stands for the orthogonal projection of f on the T-invariant functions. Also

$$\left\| \sup_N \frac{1}{\Sigma_{n\leqq N}\sigma_n}\left|\sum_{n\leqq N}\sigma_n T^n f\right|\right\|_p \leqq c_p\,\|f\|_p \quad \text{for } p>1. \tag{8.12}$$

Since $|S_N| \sim (\log N)(\log\log N)^B$, for $f \geqq 0$

$$\sup_N \frac{1}{|S_N|}\left(\sum_{n\in S_N} T^n f\right) \leqq 3 \sup_{k, N=2^{2^k}} \frac{1}{|S_N|}\sum_{n\in S_N} T^n f. \tag{8.13}$$

Thus, as a consequence of (8.13), (8.10), (8.12)

$$\left\| \sup_N \frac{1}{|S_N|}\Big(\sum_{n\in S_N} T^n f\Big)\right\|_p$$
$$\leqq 3\left\| \sup_N \frac{1}{\sum_{n\leqq N}\sigma_n}\Big(\sum_{n\leqq N}\sigma_n T^n f\Big)\right\|_p$$
$$+3\left\{\sum_{k,N=2^{2^k}}\left\|\frac{1}{\sum_{n\leqq N}\sigma_n}\Big(\sum_{n\leqq N}\sigma_n T^n f\Big)-\frac{1}{|S_n|}\Big(\sum_{n\in S_N} T^n f\Big)\right\|_p^p\right\}^{1/p}$$
$$\leqq c\,\|f\|_p+\Big(\sum_{k\geqq 1}k^{-Bp/p'}\Big)^{1/p}\|f\|_p$$
$$\leqq c\,\|f\|_p.$$

Since S satisfies the L^p-maximal inequality, the convergence a.s.

$$\frac{1}{|S_N|}\sum_{n\in S_N} T^n f \to L(f)$$

for $f\in L^p(\Omega,\mu)$ reduces to the case of bounded measurable functions. We may then restrict N of the form $[\exp(1+\varepsilon)^k]$, $k=1,2,\dots$ and the same reasoning as above together with (8.11) leads to the desired conclusion.

Remark. It is possible to satisfy Proposition 8.2(i), i.e. S verifies the mean ergodic theorem, without S satisfying a pointwise ergodic theorem on $L^q(\Omega,\mu)$ for any $q<\infty$. Examples of ergodic sequences, which do not satisfy the pointwise ergodic theorem for bounded functions, were obtained by A. Bellow and B. Weiss (using different methods).

9. Appendix 2: Commuting transformations

Assume $T_1, T_2, \dots, T_k$ are commuting measure preserving transformations on a measure space (Ω,μ). Let $p_1(n),\dots,p_k(n)$ be polynomials of $n\in Z_+$ taking integer values. Define

$$\mathscr{A}_n f=\frac{1}{n}\sum_{m=0}^{n-1} T_1^{p_1(m)}\cdots T_k^{p_k(m)} f. \tag{9.1}$$

The same method which enabled us to prove Theorems 2 and 5 also permits us to show

Theorem 6. *The maximal function* $\mathscr{M}f=\sup_{n\geqq 1}|\mathscr{A}_n f|$ *is* L^2*-bounded and* $\mathscr{A}_n f$ *converges almost surely for* f *in* $L^2(\Omega,\mu)$.

There are some natural applications of Theorem 6 where the conclusion of the usual pointwise ergodic theorem is made more precise. Consider for instance the ergodic transformation T on the 2-torus given by

$$T(x, y) = (x + y, y + \alpha)$$

where $\alpha \in R \setminus Q$. Then

$$T^n(x, y) = (x + ny + \tfrac{1}{2}n(n-1)\alpha, y + n\alpha)$$

and for φ a bounded measurable function on Π (identified with the first variable) the pointwise ergodic theorem gives

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \varphi\left(x + my + \frac{m(m-1)}{2}\alpha\right) = \int \varphi \tag{9.2}$$

almost surely with respect to the variables x, y. It follows from Theorem 6 that in fact (9.2) holds almost surely in x, for any given point $y \in \Pi$ (i.e. almost surely on every fiber). Indeed, the last statement is valid for φ a trigonometric polynomial Π (since α is an irrational) and extends to $L^2(\Pi)$ because of the boundedness of the maximal operator $\mathcal{M}f = \sup_{n\geq 1} |\mathcal{A}_n f|$ where $\mathcal{A}_n$ is given by (9.1) taking $k = 2$, $T_1 = y$-shift on Π, $T_2 = \alpha$-shift on Π, $p_1(n) = n$, $p_2(n) = n(n-1)/2$.

B. Weiss kindly pointed out to me that the commutation hypothesis on $T_1, \ldots, T_k$ is essential in order to have an ergodic theorem.

We may assume $p_1(0) = \cdots = p_k(0) = 0$ in (9.1). Notice that if $p(n)$ takes integer values for $n \in Z$, then p must have rational coefficients. If q is a common denominator of the coefficients of $p_1, \ldots, p_k$, one has for $n = mq + r$, $0 \leq r < q$

$$p_j(n) = p_j(r) + \bar{p}_j(m)$$

where $\bar{p}_j$ has integer coefficients, say

$$\bar{p}_j(m) = \sum_{1\leq t\leq s} a_{j,t} m^t \qquad (a_{jt} \in Z).$$

Hence

$$T_1^{p_1(n)} \cdots T_k^{p_k(n)} = U_1^m U_2^{(m^2)} \cdots U_s^{(m^s)} T_1^{p_1(r)} \cdots T_k^{p_k(r)}$$

denoting for $1 \leq t \leq s$

$$U_t = T_1^{a_{1,t}} \cdots T_k^{a_{k,t}}.$$

Consequently, the proof of Theorem 6 reduces to the particular case

$$p_1(m) = m, \quad p_2(m) = m^2, \quad \dots, \quad p_k(m) = m^k$$

(where k takes the value s above).

In proving the maximal inequality, the general model $(\Omega, \mu; T_1, \dots, T_k)$ is equivalent to the model $(Z^k; S_1, \dots, S_k)$ where S_j is the shift in the j-coordinate. In this case, simply define for fixed $x \in \Omega$

$$\tilde{f}(m_1, \dots, m_k) = f(T_1^{m_1} \cdots T_k^{m_k} x).$$

In the model $(Z^k, S_1, \dots, S_k)$, we have

$$\mathscr{A}_n f = f * K_n \quad \text{where } K_n = \frac{1}{n} \sum_{m=1}^{n} \delta_{(m, m^2, \dots, m^k)}.$$

The relevant exponential sums are now given by

$$\hat{K}_n(\alpha_1, \dots, \alpha_k) = \tfrac{1}{n} \sum_{m=1}^{n} e^{2\pi i (m\alpha_1 + m^2\alpha_2 + \cdots + m^k\alpha_k)}. \tag{9.3}$$

For $\theta_1, \dots, \theta_k \in [0, 1[\cap Q$ with common denominator $q < n^\delta$, define the "major box"

$$\mathscr{M}(\theta_1, \dots, \theta_k) = \{(\alpha_1, \dots, \alpha_k) \in \Pi^k \mid |\alpha_j - \theta_j| < n^{-j+\delta} (1 \leqq j \leqq k)\}.$$

Here $0 < \delta < 1$. It follows from Theorem 3, Ch. IV in [Vin] that if $\bar{\alpha} = (\alpha_1, \dots, , \alpha_k)$ is not in a major box, then

$$|\hat{K}_n(\bar{\alpha})| < n^{-\delta'} \tag{9.4}$$

for some $\delta' > 0$.

In fact, we do not need the sharper estimate of [Vin] for our purpose and the previous statement can be obtained by successive applications of Dirichlet's principle and H. Weyl's estimate Lemma 9.5, starting from the highest power k.

Lemma 9.5. *Let* $f(x) = \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_k x^k$ *and* $|\alpha_k - a/q| < 1/q^2$, $(a, q) = 1$. *Then*

$$\left| \sum_{m=1}^{n} e^{2\pi i f(m)} \right| \leqq n^{1+\varepsilon} [q^{-1} + n^{-1} + qn^{-k}]^\rho \quad \textit{where } \rho = 1/2^{k-1}$$

(cf. [Vin] p. 4)

Let now $\bar{\alpha} \in \mathcal{M}(\theta_1, \ldots, \theta_k)$ and write $\theta_j = a_j/q$, $\alpha_j = \theta_j + \beta_j$, $|\beta_j| < n^{-j+\delta}$. Write $m = qd + r$ where $0 \leqq d < n/q$ and $r = 0, 1, \ldots, q-1$. Then for $j = 1, \ldots, k$

$$\alpha_j m^j = (\theta_j + \beta_j)(qd + r)^j \in Z + \theta_j r^j + \beta_j q^j d^j + o(n^{-1+2\delta})$$

and hence

$$\frac{1}{n} \sum_{m=1}^{n} e^{2\pi i (m\alpha_1 + \cdots + m^k \alpha_k)}$$

$$= \left\{ \frac{1}{q} \sum_{r=0}^{q-1} e^{2\pi i (r\theta_1 + \cdots + r^k \theta_k)} \right\} \left\{ \frac{q}{n} \sum_{d=0}^{n/q} e^{2\pi i (\beta_1 q d + \cdots + \beta_k q^k d^k)} \right\} + o(n^{-1/2}).$$

Denote

$$\varphi(x) = a_k x^k + a_{k-1} x^{k-1} + \cdots + a_1 x$$

the rational trigonometric sum

$$S\left(\frac{\varphi(x)}{q}\right) = \sum_{r=0}^{q-1} e^{2\pi i \varphi(r)/q} \tag{9.6}$$

and

$$v_n(\bar{\beta}) = \frac{1}{n} \int_0^n e^{2\pi i (\beta_k y^k + \cdots + \beta_1 y)} dy. \tag{9.7}$$

Then, from what precedes

$$\hat{K}_n(\bar{\alpha}) = \frac{1}{q} S\left(\frac{\varphi(x)}{q}\right) v_n(\bar{\beta}) + o(n^{-1/2}). \tag{9.8}$$

Our next concern are estimates on the $S(\varphi(x)/q)$.

LEMMA.

(9.10) *If* $(a_1, \ldots, a_k, q) = 1$ *and* q *is prime, then* $|S(\varphi(x)/q)| < k\sqrt{q}$.

(9.11) *If* $(a_1, \ldots, a_k, q) = 1$, $q = p^s$ *with* p *a prime and* $s > 1$, *then*

$$\left| S\left(\frac{\varphi(x)}{q}\right) \right| \leqq c \frac{q}{p}.$$

(9.12) *If* $q = q_1 q_2 \cdots q_k$ *and* $(q_j, q_{j'}) = 1$ *for* $j \neq j'$, *then there is the identity*

$$S\left(\frac{\varphi(x)}{q}\right)=S\left(\frac{Q_1^{-1}\varphi(Q_1x)}{q_1}\right)\cdots S\left(\frac{Q_k^{-1}\varphi(Q_kx)}{q_k}\right)$$

where Q_j is defined by $q=q_jQ_j$.

(9.13) *If* $(a_1,\dots,a_k,q)=1,\quad |S(\varphi(x)/q)|\leqq c(k)q^{1-1/k}$.

(9.14) *Assume* $(a_1,\dots,a_k,q)=1$, *q has a prime factor at least equal to s and $q>s^{1+\delta}$. Then*

$$|S(\varphi(x)/q)|\leqq cqs^{-1/2-\delta'}.$$

PROOF. (9.10) is the A. Weil estimate (cf. [L–N], p. 223).

(9.11) Let $0\leqq r<p^s$, with $r=y+zp^{s-1}$ where $0\leqq y<p^{s-1}$ and $0\leqq z<p$. Then, as in (6.8)

$$\sum_{r=0}^{q-1}e^{2\pi i\varphi(r)/q}=\sum_{0\leqq y<p^{s-1}}e^{2\pi i\varphi(y)/q}\sum_{r=0}^{p-1}e^{2\pi i\varphi'(y)z/p}$$

$$=p\sum_{\substack{0\leqq y<p^{s-1}\\ p\mid\varphi'(y)}}e^{2\pi i\varphi(y)/q}.$$

It is easily checked that $\#\{0\leqq y<p^{s-1}\mid p$ divides $\varphi'(y)\}$ is bounded by $c(k)p^{s-2}$. Hence

$$|S(\varphi(x)/q)|\leqq cp^{s-1}.$$

(9.12) Is straightforward (see [Vin], Ch. III. Lemma 1).

(9.13) Is due to Hua. In fact it would suffice for our purpose to estimate $|S(\varphi(x)/q)|<cq^{1-\delta}$ for some $\delta>0$. Such an estimate is given by (9.5).

(9.14) Write $q=p^\gamma q'$ where $p\geqq s$ is prime, $\gamma\geqq 1$ and $(p,q')=1$. By (9.12) we have

$$S\left(\frac{\varphi(x)}{q}\right)=S\left(\frac{(q')^{-1}\varphi(q'x)}{p^\gamma}\right)\cdot S\left(\frac{p^{-\gamma}\varphi(p^\gamma x)}{q'}\right).$$

If $\gamma>1$, it follows from (9.1) that $|S(\varphi(x)/q)|\leqq cqp^{-1}$.

If $\gamma=1$, it follows from (9.10) and (9.13) that

$$|S(\varphi(x)/q)|\leqq cp^{1/2}(q')^{1-1/k}\leqq cqs^{1/2-1/k+1/k(1+\delta)}.$$

Proceeding as in Section 6, define for diadic values of s

$$Q_s=[s!]^{c(k)[\log s]}$$

where $c(k)$ is an integer depending on k and let further

$$\bar{\Gamma}_s = \{(a_1, \ldots, a_k, q) \mid (a_1, \ldots, a_k, q) = 1 \text{ and } q \mid Q_s\},$$

$$\Gamma_s = \bar{\Gamma}_s \setminus \Gamma_{s-1}.$$

Define for $\bar{\alpha} = (\alpha_1, \ldots, \alpha_k) \in \Pi^k$

$$\hat{L}_n(\bar{\alpha}) = v_n(\bar{\alpha}) + \sum_{s \text{ diadic}} \sum_{(a_1,\ldots,a_k,q)\in\Gamma_s} S(q, a_1, \ldots, a_k) v_n(\bar{\alpha} - \bar{\theta}) \psi_s(\bar{\alpha} - \bar{\theta})$$

$$(9.15) \qquad + \sum_{s \text{ diadic}} \sum_{\substack{(a_1,\ldots,a_k,q)\in\Gamma_s \\ q \leqq s^{1+\delta}}} S(q, a_1, \ldots, a_k) v_n(\bar{\alpha} - \bar{\theta})(\varphi_S - \psi_S)(\bar{\alpha} - \bar{\theta})$$

where v_n is given by (9.7), $\bar{\theta} = (a_1/q, \ldots, a_k/q)$,

$$S(q, a_1, \ldots, a_k) = \frac{1}{q} S\left(\frac{\varphi(x)}{q}\right) \quad \text{with } \varphi(x) = a_k x^k + \cdots + a_1 x,$$

and for $0 < \rho < 1 < \kappa$, $0 \leqq \varphi_S \leqq 1$ and $0 \leqq \psi_S \leqq 1$ are smooth functions satisfying

$$\varphi_S(\beta) = 1 \quad \text{if } |\beta| < 2^{-(s^\rho)} \quad \text{and} \quad \varphi_S(\beta) = 0 \quad \text{for } |\beta| > 2.2^{-(s^\rho)},$$

$$\psi_S(\beta) = 1 \quad \text{if } |\beta| < 2^{-(s^\kappa)} \quad \text{and} \quad \psi_S(\beta) = 0 \quad \text{for } |\beta| > 2.2^{-(s^\kappa)}.$$

Here we denote $|\beta| = \max(|\beta_1|, \ldots, |\beta_k|)$ for $\beta \in R^k$.

(9.15) Is thus a complete analogue of the formula for $\hat{L}_n(\alpha)$ given in Section 6 by (6.11), (6.12). Rewriting

$$v_n(\bar{\beta}) + \int_0^1 e^{2\pi i(n^k\beta_k x^k + n^{k-1}\beta_{k-1}x^{k-1} + \cdots + n\beta_1 x)} dx$$

it follows from van der Corput's lemma that

$$(9.16) \qquad |v_n(\bar{\beta})| \leqq c\left[1 + \sum_{1 \leqq j \leqq k} n^j |\beta_j|\right]^{-1/k}.$$

We are now in a position to repeat verbatum the proof of Lemma 6.13 to show the inequality

$$(9.17) \qquad \| \hat{K}_n - \hat{L}_n \|_\infty < c(\log n)^{-\sigma}$$

for some $\sigma > \frac{1}{2}$.

Thus, denoting again $D = \{2^k \mid k = 1, 2, \ldots\}$, the evaluation of the maximal function

$$\left\| \sup_{n\in D} |f * K_n| \right\|_{l^2(Z^k)}$$

reduces to

$$\left\| \sup_{n\in D} |f * L_n| \right\|_{l^2(Z^k)}. \tag{9.18}$$

Invoking the same argument as in Section 6, the proof of the L^2-boundedness of the maximal operator is completed by the following lemma.

LEMMA 9.19. *Let $0 \leqq \varphi \leqq 1$ be a smooth bumpfunction on R^k vanishing outside a τ-neighborhood of 0. Let R be a set of points $\bar{\theta}$ in $Q^k \cap [0,1]^k$ which can be given a common denominator Q satisfying*

$$Q\tau \ll 1.$$

Define

$$A_n f(x) = \sum_{\bar{\theta}\in R} \int v_n(\bar{\alpha} - \bar{\theta}) \hat{f}(\bar{\alpha}) e^{2\pi i \langle x, \bar{\alpha}\rangle} \varphi(\bar{\alpha} - \bar{\theta}) d\bar{\alpha}. \tag{9.20}$$

Then

$$\left\| \sup_{n\in D} |A_n f| \right\|_{l^2(Z^k)} \leqq C \| f \|_{l^2(Z^k)}. \tag{9.21}$$

PROOF. Repeating precisely the argument developed in Section 3 of this paper (in its k-dimensional version) permits us to derive formally (9.21) from the following inequality in Euclidean space:

$$\begin{aligned} &\left\| \sup_{n\in D} \left| \int v_n(\bar{\beta}) F(\bar{\beta}) \varphi(\bar{\beta}) e^{2\pi i \langle \bar{\beta}, y\rangle} d\bar{\beta} \right| \right\|_{L^2(R^k)} \\ &\leqq c \left(\int |F(\bar{\beta})|^2 \varphi(\bar{\beta})^2 d\bar{\beta} \right)^{1/2}. \end{aligned} \tag{9.22}$$

Consider the curve Γ in R^k parametrized by $\gamma(t) = (t, t^2, \dots, t^k)$ and let $n \, . \, \mu_n$ be the image measure on R^k induced by $\gamma_{[0,n]}$. Thus

$$v_n(\bar{\beta}) = \hat{\mu}_n(\bar{\beta})$$

and consequently, the left member of (9.22) equals[†]

$$\left\| \sup_{n\in D} \| (\varphi F)^{\vee} * \mu_n | \right\|_{L^2(R^k)}. \tag{9.23}$$

We know that

$$| 1 - \hat{\mu}_n(\bar{\alpha}) | \leqq c \sum_{j=1}^{k} n^j |\alpha_j| \tag{9.24}$$

and also

$$| \hat{\mu}_n(\bar{\alpha}) | \leqq c \left[1 + \sum_{j=1}^{k} n^j |\alpha_j| \right]^{-1/k}. \tag{9.25}$$

Estimate $\| \sup_{n\in D} | f * \mu_n | \|_2$ by a standard Fourier transform argument. Denote $(P_t)_{t>0}$ the standard 1-dimensional Poisson-semigroup with Fourier transform $\hat{P}_t(\lambda) = e^{-t|\lambda|}$. Define

$$K_n = P_n \otimes P_{n^2} \otimes \cdots \otimes P_{n^k}$$

on R^k. Hence

$$\sup_n | f * K_n | \leqq \sup_{t_1,\dots,t_k>0} | f * (P_{t_1} \otimes \cdots \otimes P_{t_k}) |$$

defines an L^2-bounded operator (by iteration of the 1-variable result). Next, estimate

$$\sup_{n\in D} | f * \mu_n | \leqq \sup_n | f * K_n | + \left(\sum_{n\in D} | f * (K_n - \mu_n) |^2 \right)^{1/2} \tag{9.26}$$

where

$$\| (\Sigma | f * (K_n - \mu_n) |^2)^{1/2} \|_2 = \left\{ \int_{R^k} |\hat{f}(\bar{\alpha})|^2 \left\{ \sum_{n\in D} | \hat{K}_n(\bar{\alpha}) - \hat{\mu}_n(\bar{\alpha}) |^2 \right\} d\bar{\alpha} \right\}^{1/2}$$

by Parseval's identity.

Hence it remains to show that

$$\left\| \sum_{n\in D} | \hat{K}_n(\bar{\alpha}) - \hat{\mu}_n(\bar{\alpha}) |^2 \right\|_{\infty} < c$$

[†] This is a special case of maximal functions associated to homogeneous curves, cf. [St], p. 404.

which is immediate from the fact

$$|\hat{K}_n(\bar{\alpha}) - \hat{\mu}_n(\bar{\alpha})| \leqq c \min\left\{\sum_{j=1}^{k} n^j|\alpha_j|, \left[\sum_{j=1}^{k} n^j|\alpha_j|\right]^{-1/k}\right\}$$

as a consequence of (9.24), (9.25) and since

$$\hat{K}_n(\bar{\alpha}) = e^{-[n|\alpha_1| + \cdots + n^k|\alpha_k|]}.$$

This completes the proof of Lemma 9.19 and thus the L^2-boundedness of the maximal operator associated to (9.1).

To prove the almost sure convergence of the $\mathscr{A}_n f$, we proceed as in Section 7, elaborating on what was done earlier in this section. The details are completely straightforward and left to the reader.

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